Wire Crumpling and Liouville Field Theory

http://arxiv.org/abs/0805.2896

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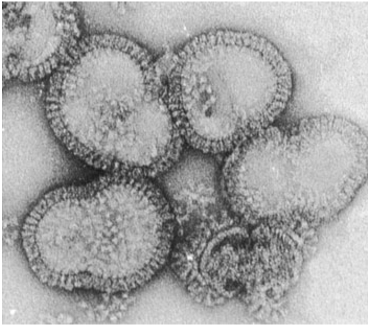
Pattern Formation, Interface Dynamics.

Crumpling: Common Phenomenon.

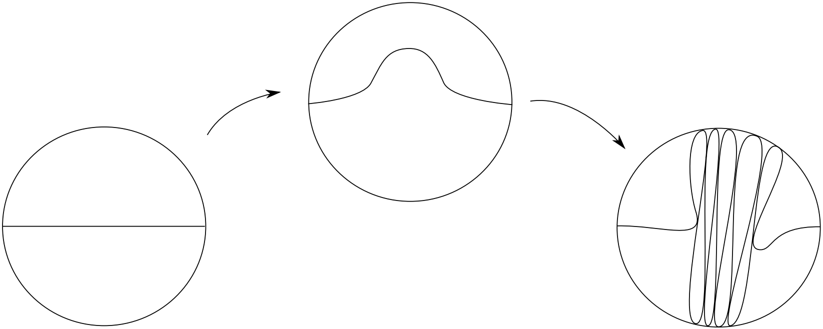


Fonte: MRC – National Institure For Medical Research – UK

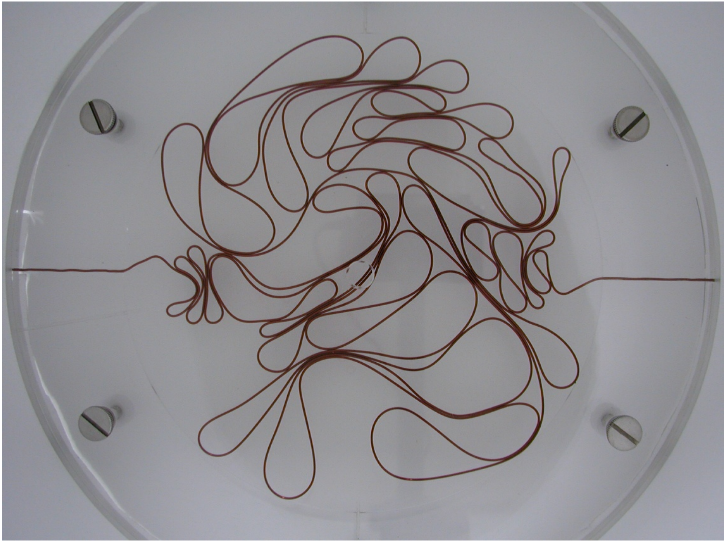
Fonte: WaterEncyclopedia.com

Crumpling of wires in 2 dimensions

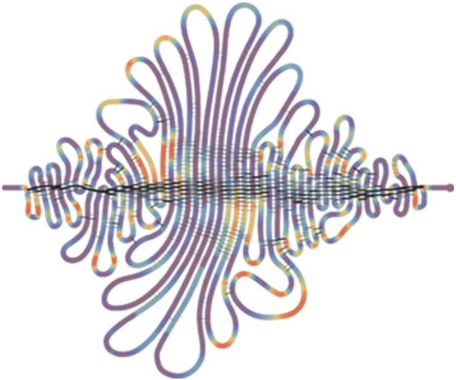
Injection Process:



Tipical Configuration



Phase Transition, Localization



N. Stoop et al., PRL 101, 094101 (2008)

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Energy dominated by wire curvature:

$$E = \frac{\alpha}{2} \int_0^L \left(\frac{d\theta}{d\ell} \right)^2 \, d\ell.$$

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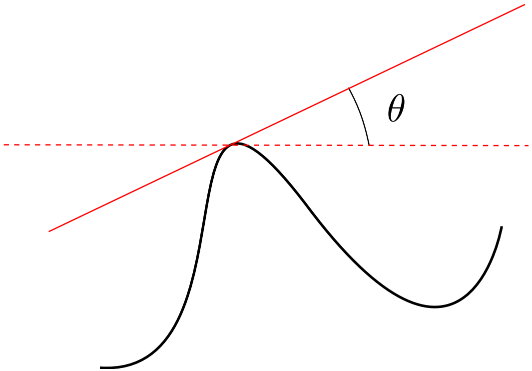

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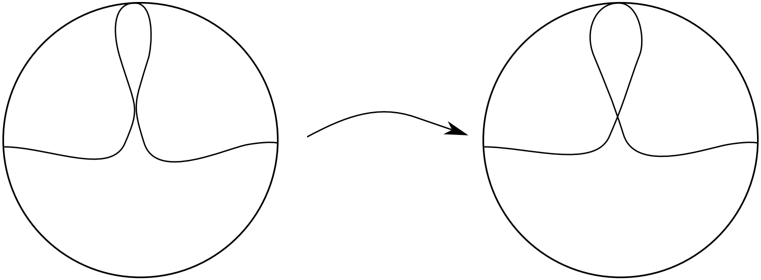
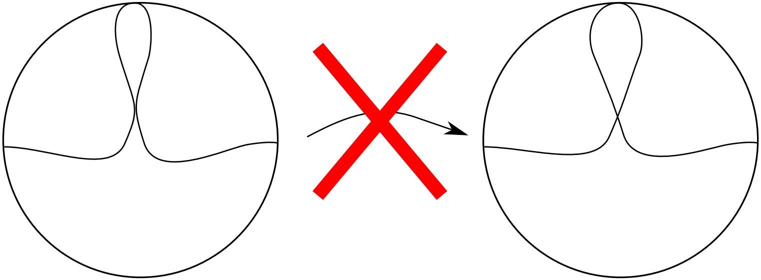
- Constant wire length;
- Fixed wire end points;
- No self-intersection;
- Wire confined inside a domain (circle).

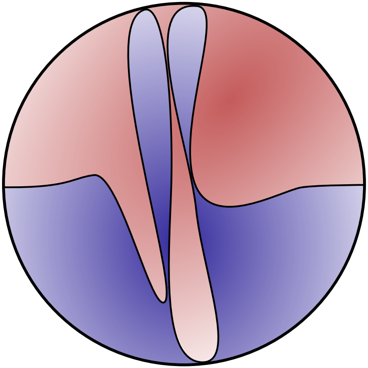
Topological Transition?



No Topological Transition!



Conformal Maps



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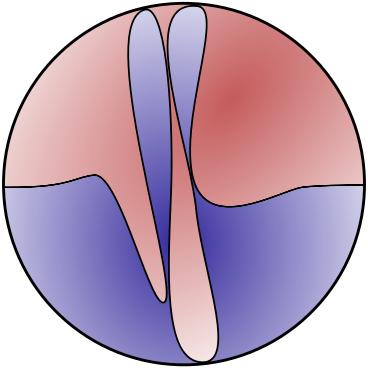

Domain uniformized by two analytical maps!

Uniformization Theorem

Every Riemann surface Σ is conformally equivalent to

- CP^1, the Riemann Sphere, or
- H, the Poincaré upper-half plane, or
- X, a quotient of H by a discreet subgroup $\Gamma \subset \mathsf{SL}(2,\mathbb{R})$ acting as a Möbius transformation.

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$$ds^2 = -\frac{16}{\mu}\frac{\partial A(z)\bar{\partial}B(\bar{z})}{(A(z)-B(\bar{z}))^2}dz\,d\bar{z}.$$

Classical Uniformization

Hence, for a given 2d and connected domain Σ, the solution of the equation

$$\partial\bar{\partial}\phi = \frac{\mu}{16}e^{2\phi},$$

of the form

$$\phi(z,\bar{z}) = \frac{1}{2}\log\left[-\frac{16}{\mu}\frac{\partial A(z)\bar{\partial}B(\bar{z})}{(A(z)-B(\bar{z}))^2}\right],$$

gives us a conformal map $A(z)$ (e $B(\bar{z}) = [A(z)]^*$) whose image is X (seen as a part of H).

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By writing the problem of wire crumpling in the Liouville field language, we can deal with the domain and no self-intersection constraints.

Relation with Gravity

Take Einstein-Hilbert

$$S_{EH} = \frac{1}{16\pi G_N} \int d^n x \sqrt{g}(R - 2\Lambda),$$

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$$G_N = \frac{(n-2)}{8}\gamma^2, \quad \Lambda = \frac{(n-2)}{8}\lambda$$

results in the Liouville action for the conformal mode of the metric $g_{ab} = e^{\gamma\phi}\hat{g}_{ab}$:

$$S_L = \frac{1}{8\pi} \int d^2 x \sqrt{\hat{g}} \left[(\hat{\nabla}\phi)^2 + Q\hat{R}\phi + \frac{\lambda}{\gamma^2} e^{\gamma\phi} \right], \qquad Q = \frac{2}{\gamma} + \gamma.$$

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Limit actually accomplished in various quantum versions of gravitational theories, including (super)strings.

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We want to see $z(t), t \in \mathbb{R}$ as the conformation of the wire. As in general we have $\partial_w z = e^{-v+iu}$, can write the boundary conditions in terms of u and v:

$$u(w, \bar{w} = 1/w) = \psi(w) + i \log w$$
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Combining both of them:

$$e^{-v} = \partial_n v + 1 = \partial_n v + K, \text{ quando } w = 1/\bar{w}.$$

Boundary conditions for ν follow from a variational principle:

$$\mathscr{S} = \frac{1}{2\pi}\int_D d^2z\,\partial\nu\bar{\partial}\nu + \frac{1}{4\pi}\oint_{\partial D} d\ell\,(\nu K + e^{-\nu});$$

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Introducting the free energy:

$$\mathscr{F}(\alpha, \mu) = \frac{\alpha}{2} \int_{-1}^{1} \mathrm{d}t\, (\dot{\mathrm{u}})^2 \mathrm{e}^{\mathrm{v}} + \mu \int_{-1}^{1} \mathrm{d}t\, \mathrm{e}^{-\mathrm{v}},$$

we have a prescription to the calculation of expectation values for the energy and wire length:

$$\langle \mathrm{E} \rangle = \frac{\partial}{\partial \alpha} \langle \mathrm{e}^{-\mathscr{F}} \rangle \Big|_{\alpha=\mu=0} \qquad \langle \mathrm{L} \rangle = \frac{\partial}{\partial \mu} \langle \mathrm{e}^{-\mathscr{F}} \rangle \Big|_{\alpha=\mu=0}$$

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- Given a solution $\nu(w, \bar{w})$ in the domain, it is harmonic except on the wire, when $w = t \in \mathbb{R}$. With this we can construct analytical maps from the upper and lower semi-circles, whose images are connected regions. The curve which bounds both regions will be then simple.

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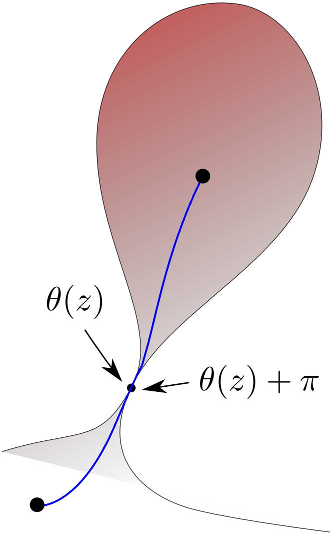
- Constraint at the initial and final endpoints ($t = \pm 1$) can be implemented by computing field correlations at those points.

Description of the Loop



The Schwarz function

Parametrizing the wire by $t = z$, and calling $\bar{z}(t(z)) = S(z)$:

$$\frac{dz}{d\ell} = \frac{1}{\sqrt{S'(z)}}, \qquad k = \frac{d\theta}{d\ell} = \frac{i}{2S'}\frac{dS'}{d\ell} = \frac{i}{2S'}\frac{dz}{d\ell}\frac{dS'}{dz} = \frac{i}{2}\frac{S''}{(S')^{3/2}}$$

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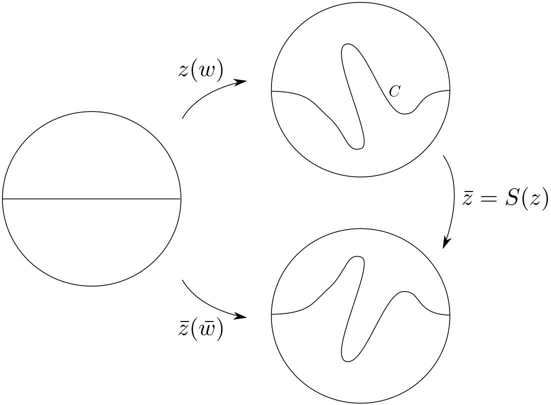

Extremizing the energy with the constraints of fixed endpoints results

$$\frac{1}{S'}\{S;z\} + \frac{\lambda}{\alpha}\sqrt{S'} - \frac{\lambda}{\alpha}\frac{1}{\sqrt{S'}} = 0,$$

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Changing variables $S'(z) = e^{2\gamma\phi(z)}$, the equation can be written as

$$T + V = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2\gamma}\partial^2\phi - \frac{\lambda}{4\gamma^2}e^{\gamma\phi}\left[1 - e^{2\gamma\phi}\right] = 0$$

which is the "Gravitational Ward Identity" of 2d-QG.

Some methods from Conformal Field Theories

T implements conformal transformations at the field level, in particular, scaling relations. Then we can study the asymptotic limit of the observables:

$$E = -\frac{\alpha}{4}\int_C dz\,\sqrt{S'}\,\frac{(S'')^2}{(S')^3} = -\alpha\gamma^2\int_C dz\,(\partial\phi)^2 e^{-\gamma\phi},$$

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The term proportional to the second derivative $\partial^2\phi$ can be understood as a coupling to the curvature of the base space of the form $Q\sqrt{g}\phi\hat{R}$, $Q = 2/\gamma$. This contributes to the anomalous dimensions of the observables via the Coulomb Gas formalism.

Ward Identities in 2-d

Action of T over the fields is determined by the Operator Product Expansion, computed for free fields via Wick's theorem:

$$T(z)\Phi_h(w) = \frac{h\Phi_h(w)}{(z-w)^2} + \frac{\partial\Phi_h(w)}{z-w} + \text{regular}$$

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Operators of the form $V_\beta = e^{\beta\phi}$ are called vertex operators, with $h[V_\beta] = \beta^2/2$ well defined. Owing to the OPE

$$\partial^2\phi(z)V_\beta(w) \backsim \frac{\beta}{(z-w)^2}V_\beta(w),$$

the effect of the second derivative term is to correct h to:

$$h[V_\beta] = \frac{\beta}{2\gamma} - \frac{\beta^2}{2}.$$

The γ factor is determined by the requirement that both terms composing the constraint have the same dimension:

$$\mathsf{h}[\mathsf{e}^{\gamma\phi}] = \frac{1-\gamma^2}{2}, \quad \mathsf{h}[\mathsf{e}^{3\gamma\phi}] = \frac{3-9\gamma^2}{2} \to \gamma = 1/2.$$

Fator agrees with the value of Q obtained in the last section. Then the scaling dimensions of E and L are:

$$\mathsf{h}[\mathsf{L}] = -5/8, \qquad \mathsf{h}[\mathsf{E}] = 2(\mathsf{h}[\mathsf{e}^{-\frac{1}{2}\gamma\phi}] + 1) - 1 = 7/16.$$

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As there is only one free dimensionful parameter in the theory, the maximum amount of length L_{crit}, we can then relate both observables:

Conclusions

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- Energy grows with a negative exponent: localization.
- Testable exponent.
- New application of Liouville Field Theory.