Solução da Terceira lista de relatividade geral

Universidade Federal de Pernambuco

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Professor: Bruno Carneiro Cunha
Monitor: Cesar Leonardo
Problema 1.

$ds^2 = \Omega^2$ (- $dt^2 + dx^2$)

$\Gamma^t_{tt} = -\frac{\Omega^{-2}}{2}$ (- $\partial_t \Omega^2$) $= \partial_t \log\Omega$
$\Gamma^t_{rt} = -\frac{\Omega^{-2}}{2}$ (- $\partial_r \Omega^2$) $= \partial_r \log\Omega = \Gamma^t_{tr}$
$\Gamma^t_{rr} = -\frac{\Omega^{-2}}{2}$ (- $\partial_t \Omega^2$) $= \partial_t \log\Omega$
$\Gamma^r_{tt} = \frac{\Omega^{-2}}{2}$ (+ $\partial_r \Omega^2$) $= \partial_r \log\Omega$
$\Gamma^r_{tr} = \frac{\Omega^{-2}}{2}$ ($\partial_t \Omega^2$) $= \partial_t \log\Omega = \Gamma^r_{rt}$
$\Gamma^r_{rr} = \frac{\Omega^{-2}}{2}$ ($\partial_r \Omega^2$) $= \partial_r \log\Omega$

$R_{trt}{}^r = - \partial_t \Gamma^r_{rt} + \partial_r \Gamma^r_{tt} + \Gamma^\alpha_{tt}\Gamma^r_{\alpha r} - \Gamma^\alpha_{rt}\Gamma^r_{\alpha t}$

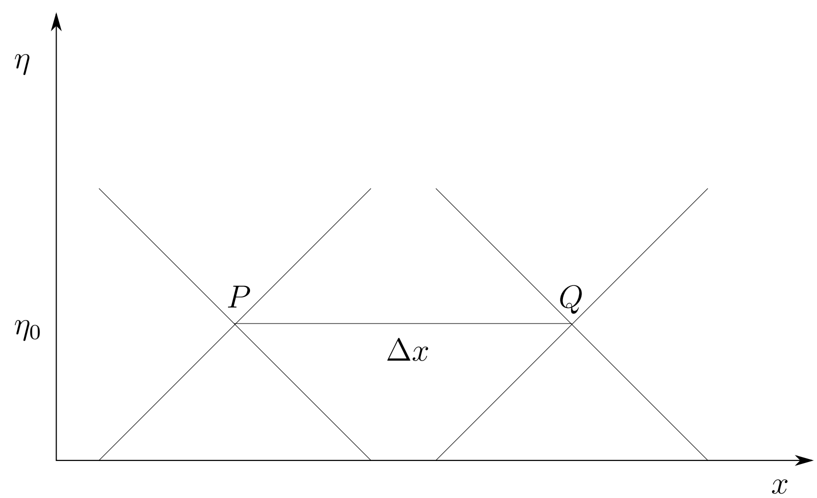
$= - \partial_t{}^2 \log\Omega + \partial_r{}^2 \log\Omega + \Gamma^r_{tt}\Gamma^r_{rr} - \Gamma^r_{rt}\Gamma^r_{rt} + \Gamma^t_{tt}\Gamma^r_{tr} - \Gamma^t_{rt}\Gamma^r_{tt}$
$= (- \partial_t{}^2 + \partial_r{}^2)\log\Omega + (\partial_r \log\Omega)^2 - (\partial_t \log\Omega)^2 -$
$- (\partial_r \log\Omega)^2 + (\partial_t \log\Omega)^2$
$R_{trt}{}^r = (- \partial_t{}^2 + \partial_r{}^2)\log\Omega \Rightarrow R_{trtr} = R_{rtrt} = \Omega^2(- \partial_t{}^2 + \partial_r{}^2)\log\Omega$
$R_{tt} = (- \partial_t{}^2 + \partial_r{}^2)\log\Omega$
$R_{rr} = - (- \partial_t{}^2 + \partial_r{}^2)\log\Omega$
$R = 2\Omega^{-2}(- \partial_t{}^2 + \partial_r{}^2)\log\Omega$
$R_{abcd} = \frac{R}{2} (g_{ac}g_{bd} - g_{ad}g_{bc})$

problema 2.

ds^2 = $-dt^2$ + t^{2q} (dx^2 + dy^2 + dz^2)

$d\eta = t^{-q}dt \Rightarrow \eta = \int \frac{dt^{'}}{t'^{q}} = \frac{t^{1-q}}{1-q}$ (*)

ds^2 = t^{-q} (- $d\eta^2$ + dx^2 + dy^2 + dz^2) , onde os termos entre parenteses pertencem a um espaço de Minkowski com η definido-positivo.



Desde que $0 \leq q < 1$, o integral para η converge uniformemente para $t \rightarrow \infty$, assim , teremos horizontes no passado e no futuro.

Por exemplo:

$a(t) = \sqrt{1 + t^{2q}}$, etc.

Problema 3.

$ds^2 = - f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2$

$\Gamma^t_{tt} = 0$
Γ^t_{rt} = -$\frac{f^{-1}}{2}$ (-∂_rf) = + $\frac{f'}{2f}$; Γ^t_{rr} = 0
$\Gamma^t_{r\phi}$ = 0 ; $\Gamma^t_{\phi\phi}$ = 0 ; Γ^t_{rr} = 0
Γ^r_{tt} = $\frac{f}{2}$ (+∂_rf) = + $\frac{1}{2}$ f f'
Γ^r_{rt} = $\frac{f}{2}$ ∂_rf^{-1} = - $\frac{1}{2}\frac{f'}{f}$
$\Gamma^r_{\phi\phi}$ = $\frac{f}{2}$ (- ∂_rr^2) = -rf
Γ^r_{tr} = 0 ; $\Gamma^r_{\phi r}$ = 0 ; $\Gamma^r_{t\phi}$ = 0
$\Gamma^\phi_{t\phi} = \Gamma^\phi_{tt} = \Gamma^\phi_{rr} = \Gamma^\phi_{\phi\phi} = \Gamma^\phi_{rt} = 0$
$\Gamma^\phi_{r\phi}$ = $\frac{r^{-2}}{2}\partial_r$r^2 = $\frac{1}{r}$

R$_{trt}{}^r$ = -$\partial_t\Gamma^r_{rt}$ + $\partial_r\Gamma^r_{tt}$ + $\Gamma^\alpha_{tt}\Gamma^r_{\alpha r}$ - $\Gamma^\alpha_{rt}\Gamma^r_{\alpha t}$

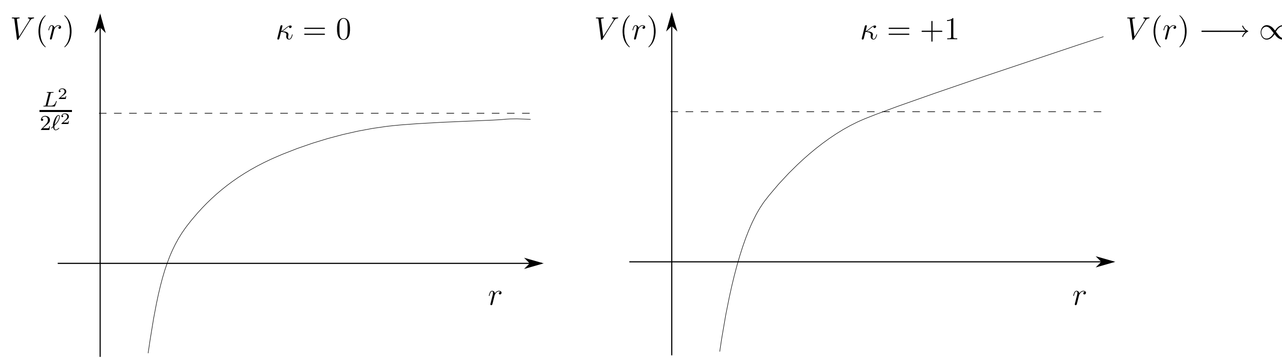
= +$\frac{1}{2}$f f" + $\frac{1}{2}$f'^{2} + $\frac{1}{2}$f f' (-$\frac{1}{2}\frac{f'}{f}$) - ($\frac{1}{2}\frac{f'}{f}$) ($\frac{1}{2}$f f') = +$\frac{1}{2}$f f"
R$_{trt}{}^\phi$ = R$_{rtr}{}^\phi$ = R$_{\phi r\phi}{}^t$ = 0
R$_{t\phi t}{}^\phi$ = - $\partial_t\Gamma^\phi_{\phi t}$ + $\partial_\phi\Gamma^\phi_{tt}$ + $\Gamma^\alpha_{tt}\Gamma^\phi_{\alpha\phi}$ - $\Gamma^\alpha_{t\phi}\Gamma^\phi_{\alpha t}$ = $\Gamma^r_{tt}\Gamma^\phi_{r\phi}$ = ($\frac{1}{2}$f f')$\frac{1}{r}$
R$_{r\phi r}{}^\phi$ = - $\partial_r\Gamma^\phi_{\phi r}$ + $\partial_\phi\Gamma^\phi_{rr}$ + $\Gamma^\alpha_{rr}\Gamma^\phi_{\alpha\phi}$ - $\Gamma^\alpha_{r\phi}\Gamma^\phi_{\alpha r}$ =
= + $\frac{1}{r^2}$ + (- $\frac{1}{2}\frac{f'}{f}$) $\frac{1}{r}$ - $\frac{1}{r^2}$ = - $\frac{1}{2r}\frac{f'}{f}$
R$_{trtr}$= g$_{rr}$R$_{trt}{}^r$ = $\frac{1}{2}$f" = $\frac{1}{l^2}$ = - $\frac{1}{l^2}$ (g$_{tt}$g$_{rr}$ - g$_{tr}$g$_{tr}$)
R$_{t\phi t\phi}$ = g$_{\phi\phi}$R$_{t\phi t}{}^\phi$ = $\frac{r}{2}$ f f' = $\frac{r^2}{l^2}$ f = - $\frac{1}{l^2}$ (g$_{tt}$g$_{\phi\phi}$ - g$_{t\phi}$g$_{t\phi}$)
R$_{r\phi r\phi}$ = g$_{\phi\phi}$R$_{r\phi r}{}^\phi$ = - $\frac{r}{2}$ $\frac{f'}{f}$ = - $\frac{r^2}{l^2}$ $\frac{1}{f}$ = - $\frac{1}{l^2}$ (g$_{rr}$g$_{\phi\phi}$ - g$_{t\phi}$g$_{r\phi}$)
R$_{ab}$ = - $\frac{2}{l^2}$ g$_{ab}\rightarrow$ curvatura escalar = - $\frac{6}{l^2}$

Problema 4.

ds^2 = - (- M + $\frac{r^2}{l^2}$) dt^2 + $\frac{dr^2}{(-M+\frac{r^2}{l^2})}$ + r^{2}dϕ^2
t^a = ($\frac{\partial}{\partial t}$)^a ; ϕ^a = ($\frac{\partial}{\partial\phi}$)^a,
com v^a geodésico e normalizado teremos :
t$_a$v^a = E , - (- M + $\frac{r^2}{l^2}$) t = E
ϕ_av^a = l ; r^{2}t = L

Geodésicas

- K = - $\frac{E^2}{-M+\frac{r^2}{l^2}}$ + $\frac{r^2}{-M+\frac{r^2}{l^2}}$ + r^2 $\frac{L^2}{r^4}$
E^2 = + K (- M + $\frac{r^2}{l^2}$) + $\frac{L^2}{r^2}$ (- M + $\frac{r^2}{l^2}$) + r$^{\cdot 2}$
$\frac{E^2}{2}$ = ($\frac{K}{2}$ + $\frac{L^2}{2r^2}$) (- M + $\frac{r^2}{l^2}$) + $\frac{r^2}{2}$
A primeira parcela do lado direito corresponde ao potencial efetivo.

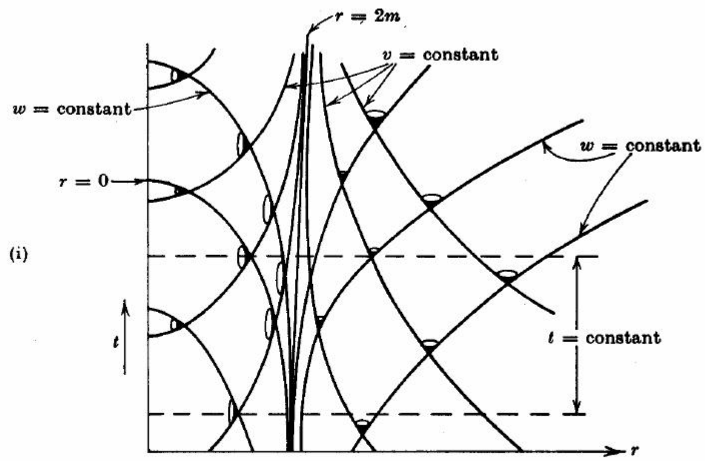


$\frac{dr}{d\tau}$ = $\frac{dr}{dt}\frac{dt}{d\tau}$ = t$\frac{dr}{dt}$ = - $\frac{E}{-M+\frac{r^2}{l^2}}\frac{dr}{dt}$

$\frac{E^2}{2}$ = ($\frac{K}{2}$ + $\frac{L^2}{2r^2}$) (- M + $\frac{r^2}{l^2}$) - $\frac{E^2}{2(-M+\frac{r^2}{l^2})^2}$ ($\frac{dr}{dt}$)^2

Note que , para K = 0 , tipo-nulo especifica cone de luz , o resto segue a região futuro.

(1 - $\frac{L^2}{E^2 r^2}$ (-M+$\frac{r^2}{l^2}$)^2) (-M+$\frac{r^2}{l^2}$) = ($\frac{dr}{dt}$)^2



$r_0 = \sqrt{M}$ l e $\phi = \phi_0$ é uma geodésica.
Com L = 0 e $(\frac{dr}{dt})^2 = 0$